

A power consensus algorithm for DC microgrids [★]

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Abstract

A novel power consensus algorithm for DC microgrids is proposed and analyzed. DC microgrids are networks of transmission lines connected to sources and loads. They are represented by differential-algebraic equations connected over an undirected weighted graph that models the electrical circuit. A second graph represents the communication network over which the source nodes exchange information about the instantaneous powers, which is used to adjust the injected current accordingly. This gives rise to a nonlinear consensus-like system of differential-algebraic equations that is analyzed via Lyapunov functions inspired by the physics of the system. We establish convergence to the set of equilibria consisting of weighted consensus power vectors as well as preservation of the geometric mean of the source voltages. The results apply to networks with constant impedance, constant current and constant power loads.

Key words: DC microgrids, Power sharing, Distributed control, Nonlinear consensus, Lyapunov stability analysis

1 Introduction

The proliferation of renewable energy sources and storage devices that are intrinsically operating using the DC regime is stimulating interest in the design and operation of DC microgrids, which have the additional desirable feature of preventing the use of inefficient power conversions at different stages. These DC microgrids might have to be deployed in areas where an AC microgrid is already in place, creating what is called a hybrid microgrid [1], for which rigorous analytical studies are still in their infancy. Furthermore, the envisioned future in which power generation is far away from the major consumption sites raises the problem of how transmitting power with low losses, a problem for which High Voltage Direct Current (HVDC) networks perform comparatively better than AC networks. Finally, also mobile grids on ships, aircrafts, and trains are based on a DC architecture.

With DC and hybrid microgrids, as well as HVDC networks, on the rise, we need to develop a deeper system-theoretic understanding of the system theoretic proper-

ties of this interesting class of dynamical networks. In this paper we propose and analyse a control algorithm for a DC microgrid that enforces power sharing among the different power sources.

1.1 Literature review

The literature on DC microgrids is rapidly growing. We summarize below the contributions that share a systems and control theoretic point of view on these networks. The work [2] relies on a cooperative control paradigm for dc microgrids to replace the conventional secondary control by a voltage and a current regulator. In [3] a voltage droop controller for DC microgrids inspired by frequency droop in AC power networks is analyzed and a secondary consensus control strategy is added to prevent voltage drift and achieve optimal current injection. The paper [4] models the DC microgrid via the Brayton-Moser equations and uses this formalism to show that with the addition of a decentralized integral controller voltage regulation to a desired reference value is achieved. Other schemes achieving desirable power sharing properties are proposed but no formal analysis is provided. In [5], a secondary consensus-based control scheme for current sharing and voltage balancing in DC microgrids is designed in a Plug-and-Play fashion to allow for the addition or removal of generation units. A distributed control method to enforce power sharing among a cluster of dc microgrids is proposed in [6]. Other work has focused on the challenges in the stability analysis of DC microgrids using consensus-like algorithms due to the interaction be-

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tween the communication network and the physical one [7]. Finally, feasibility of the nonlinear algebraic equations in DC power circuits is studied by [8], [9], and [10].

A closely related research area is that of multi-terminal HVDC transmission systems. The paper [11] focuses on cooperative frequency control for these networks. In [12] distributed controllers that keep the voltages close to a nominal value and guarantee a fair power sharing are considered, whereas passivity-based decentralized PI control for the global asymptotic stabilisation of multi-terminal high-voltage is studied in [13]. The paper [14] studies feasibility and power sharing under decentralized droop control. We refer to [15, Chapter 4] for an annotated bibliography of HVDC transmission systems.

1.2 Main contribution

The paper focuses on a new control algorithm to stabilize a DC microgrid under different load characteristics while achieving power sharing among the sources. Our controller is enabled by communicating the instantaneous source power measurements among neighboring source nodes, averaging these measurements and setting the voltage at the source terminals accordingly. An additional feature of the algorithm is that the geometric average of the source voltages is preserved.

The system dynamics present interesting features. In fact, by averaging the power measurements that the sources communicate amongst each other, the system dynamics becomes an intriguing combination of the physical network (the weighted Laplacian of the electrical circuit appearing in the power measurements) and the communication network (over which the information about the power measurements is exchanged). “ZIP” (constant impedance, constant current and constant power) loads introduce algebraic equations in the system’s dynamics, adding additional complexity and nonlinearities.

To analyze this system of nonlinear differential-algebraic equations without going through a linearization of the dynamics, Lyapunov based arguments become very convenient. The Lyapunov functions in this case are constructed starting from the power dissipated in the network that is further shaped to take into account the specifics of the dynamics. The presence of the loads, which shift the equilibrium of interest, is taken into account by the so-called Bregman storage functions [16]. The level sets of the Lyapunov functions are used to estimate the excursion of the state response of these systems and therefore, combined with the preservation of the geometric average of the source voltages, can be used to obtain an estimate of the voltage at steady state.

Power sharing algorithms have been first suggested by [17] for network-reduced AC microgrids whose voltage

dynamics show similar features as in DC grids. In this paper we show that a similar idea can be adopted also for network preserved DC microgrids. The novelties of this contribution with respect to [17] are the different dynamics of the system under study, the explicit consideration of algebraic equations in the model and the use of Lyapunov arguments to prove the main results.

1.3 Paper organization

The model of the DC microgrid is introduced in Section 2. The power consensus algorithm is introduced in Section 3. The analysis of the closed-loop system is first carried out in Section 4 in the case of constant current loads. This relatively simpler case allows us to set the ground for the analysis in the case of constant impedance loads (Section 5) and for the constant power load case (Section 6). A numerical test of the algorithm is discussed in Section 7. Conclusions are drawn in Section 8.

1.4 Notation

Given a vector v , the symbol $[v]$ represents the diagonal matrix whose diagonal entries are the components of v . The notation $\text{col}(v_1, v_2, \dots, v_n)$, with v_i scalars, represents the vector $[v_1 \ v_2 \ \dots \ v_n]^T$. If v_i are matrices having the same number of columns, then $\text{col}(v_1, v_2, \dots, v_n)$ denotes the matrix $[v_1^T \ v_2^T \ \dots \ v_n^T]^T$. The symbol $\mathbf{1}_n$ represents the n -dimensional vector of all 1’s, whereas $\mathbf{0}_{m \times n}$ is the $m \times n$ matrix of all zeros. When the size of the matrix is clear from the context the index is omitted. The $n \times n$ identity matrix is represented as \mathbb{I}_n . Given a vector $v \in \mathbb{R}^n$, the symbol $\mathbf{ln}(v)$ denotes the element-wise logarithm, i.e., the vector $[\ln(v_1) \ \dots \ \ln(v_n)]^T$.

2 DC resistive microgrid

The DC microgrid is modeled as an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} := \{1, 2, \dots, n\}$ the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. The edges represent the transmission lines of the microgrid, which we assume here to be resistive. Associated to each edge is a weight modeling conductance (or reciprocal resistance) $1/r_k > 0$, with $k \in \mathcal{E}$. The set of nodes is partitioned into the two subsets of n_s sources \mathcal{V}_s and n_l loads \mathcal{V}_l , with $n_s + n_l = n$.

The current-potential relation in a resistive network is given by the identity $I = B\Gamma B^T V$, with B the incidence matrix of \mathcal{G} and $\Gamma = \text{diag}\{r_1^{-1}, \dots, r_m^{-1}\}$ the diagonal weight matrix. Considering the partition of the nodes in sources and loads, the relation rewrites as

$$\begin{bmatrix} I_s \\ I_l \end{bmatrix} = \begin{bmatrix} B_s \Gamma B_s^T & B_s \Gamma B_l^T \\ B_l \Gamma B_s^T & B_l \Gamma B_l^T \end{bmatrix} \begin{bmatrix} V_s \\ V_l \end{bmatrix} =: \begin{bmatrix} Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} \end{bmatrix} \begin{bmatrix} V_s \\ V_l \end{bmatrix}. \quad (1)$$

Being principal submatrices of a Laplacian of a connected undirected graph, both Y_{ss} and Y_{ll} are positive definite, a property that will be used throughout the paper.

3 Power consensus controllers

We propose controllers that force the different sources to share the total power injection [17]. For this purpose, a communication network is deployed to connect the source nodes, through which the controllers exchange information about the instantaneous injected powers. This communication network is modelled as an undirected unweighted graph $(\mathcal{V}_c, \mathcal{E}_c)$, where $\mathcal{V}_c = \mathcal{V}_s$. Associated with the communication graph is the $n_s \times n_s$ Laplacian matrix $L_c = D_c - A_c$, where D_c is the degree matrix and A_c is the adjacency matrix of the communication graph. Note that the nodes of the communication network (but not necessarily the edges) coincide with the source nodes of the microgrid. For each node $i \in \mathcal{V}_s$, the set $\mathcal{N}_{c,i} = \{j \in \mathcal{V}_s : \{i, j\} \in \mathcal{E}_c\}$ represents the neighbors connected to node i via the communication graph.

Controllers. The proposed controllers are of the form

$$C_i(V_i)\dot{V}_i = -I_i + u_i, \quad i \in \mathcal{V}_s, \quad (2)$$

where

$$C_i(V_i) = V_i^{-2} D_{ci}^{-1} C_i^2, \quad i \in \mathcal{V}_s \quad (3)$$

can be interpreted as a nonlinear capacitance, $C_i > 0$ is a positive parameter of suitable units such that $C_i(V_i)$ actually has the units of a capacitance, I_i is the injected current at node $i \in \mathcal{V}_s$ as defined in (1), and the term

$$u_i = V_i^{-1} D_{ci}^{-1} C_i \sum_{j \in \mathcal{N}_{c,i}} C_j^{-1} P_j, \quad i \in \mathcal{V}_s \quad (4)$$

represents an ideal current source that is controlled as a function of the local voltage V_i and the injected power $P_j = V_j I_j$ at the neighboring node sources $j \in \mathcal{N}_{c,i}$.

The dynamic controllers (2)–(4) are initialised at positive values of the voltage, that is $V_i(0) > 0$ for all $i \in \mathcal{V}_s$. It will be made evident in later sections that these controllers makes the positive orthant $\mathbb{R}_{>0}^{n_s}$ positively invariant, thus showing that the positivity of the initial source voltages yields positivity of these variables for all $t \geq 0$.

Remark 1 (Circuit interpretation) *The control algorithm has the circuit interpretation given in Fig. 1. Comparing with [4, (4)], the ideal current source u_i can be generated also by a voltage source with value v_i in series with a resistance r_i provided that $v_i = r_i u_i + V_i$. Finally, the dynamic droop controller in [3] corresponds in our notation to a constant capacitance C_i and current source u_i .*

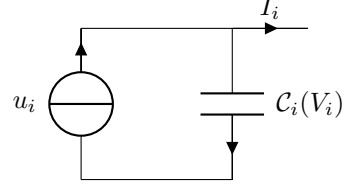


Fig. 1. A circuit interpretation of the controller (2).

Remark 2 (Digital implementation) *The control algorithm above need not be implemented analogically. In its digital implementation, information regarding the injected powers of neighbouring sources, P_j , $j \in \mathcal{N}_{c,i}$, are broadcasted and processed along with the current measurement I_i to compute the source voltage value V_i applied at the source terminals.*

Multiplying both sides of (2) by $V_i^2 D_{ci} C_i^{-1}$, one arrives at the closed-loop system

$$\begin{aligned} C_i \dot{V}_i &= -V_i D_{ci} C_i^{-1} P_i + V_i \sum_{j \in \mathcal{N}_{c,i}} C_j^{-1} P_j \\ &= V_i \sum_{j \in \mathcal{N}_{c,i}} (C_j^{-1} P_j - C_i^{-1} P_i), \quad i \in \mathcal{V}_s, \end{aligned} \quad (5)$$

that is, the voltage at the source terminal is updated according to a weighted power consensus algorithm scaled by the voltage. For interpretation purposes, we write (5) as

$$\frac{d}{dt} C_i \ln(V_i) = \sum_{j \in \mathcal{N}_{c,i}} (C_j^{-1} P_j - C_i^{-1} P_i), \quad i \in \mathcal{V}_s.$$

In a classic power system analysis [18], the term $C_i \ln(V_i)$ is the natural energy representation of a constant power source of value C_i . The interpretation of the closed-loop system (5) is then that this constant power source is adapted according to a power consensus algorithm.

Remark 3 (Proportional power sharing) *Provided that $V_i \neq 0$ (a property that will be established in the next sections), equation (5) shows that at steady state the algorithm achieves proportional power sharing according to the C_i ratios, namely*

$$\frac{P_j}{C_j} = \frac{P_i}{C_i}, \quad \forall i, j \in \mathcal{V}_s. \quad (6)$$

A detailed characterisation of the steady state value of the power signals is given in the next section (Lemma 1).

Remark 4 (Current consensus algorithm) *It is interesting to compare the algorithm (5) with the distributed current consensus filter proposed in [3]. The latter takes*

the form

$$C_i \dot{V}_i = -(I_i - I_i^*) - p_i \quad (7)$$

$$D_i \dot{p}_i = -(I_i - I_i^*) - p_i - \sum_{j \in \mathcal{N}_{c,i}} (\alpha_j p_j - \alpha_i p_i), \quad (8)$$

where $i \in \mathcal{V}_s$, p_i is a control variable, $D_i > 0$ a time constant, I_i^* a current injection setpoint, whereas the positive constants α are the coefficients of a quadratic cost function aiming at enforcing optimal current injections. In (8) the consensus term forces the control variables p_i to converge to the optimal current injection, while the integral action asymptotically provides the required feed-forward input needed to compensate for the unmeasured constant current load.

Remark 5 (DAPI power sharing control) A possibly more simplistic and obvious power sharing controller based on a distributed averaging proportional integral law (for which no analysis is provided) is

$$\begin{aligned} C_i \dot{V}_i &= -I_i + p_i \\ D_i \dot{p}_i &= I_i - p_i - \sum_{j \in \mathcal{N}_{c,i}} (C_j^{-1} V_j p_j - C_i^{-1} V_i p_i), \quad i \in \mathcal{V}_s \end{aligned} \quad (9)$$

where p_i is in units of currents. Any steady state of this controller would guarantee for all $i \in \mathcal{V}_s$ that $\dot{V}_i = 0$ is in steady state and the vector of power injections $C_s^{-1} [V_s] p$ has all identical entries (power sharing). Numerical results (see Section 7) show that (5) and (9) perform similarly and in the rest of the paper we focus on the analysis of (5).

Loads. Depending on the particular load models, the term I_l in (1) takes different expression and will henceforth be denoted as $I_l(V_l)$ to stress the functional dependence on the load voltages. In this paper we consider the following:

- (i) constant current loads: $I_l(V_l) = I_l^* \in \mathbb{R}_{<0}^{n_l}$,
- (ii) constant impedance: $I_l(V_l) = -Y_l^* V_l$, with $Y_l^* > 0$ a diagonal matrix of load conductances, and $V_l = \text{col}(V_{n_s+1}, \dots, V_{n_s+n_l})$, and
- (iii) constant power: $I_l(V_l) = [V_l]^{-1} P_l^*$, with $P_l^* \in \mathbb{R}_{<0}^{n_l}$.

To refer to the equilibria set of the systems corresponding to the three load cases above, we will use the indices “I”, “Z” and “P” respectively.

Bearing in mind (1), (5), and vectorizing the expressions to avoid cluttered formulas, the closed-loop system is

$$\begin{bmatrix} C_s \dot{V}_s \\ -I_l(V_l) \end{bmatrix} = - \begin{bmatrix} [V_s] L_c C_s^{-1} P_s \\ B_l \Gamma B^T V \end{bmatrix}, \quad (10)$$

where $V_s = \text{col}(V_1, \dots, V_{n_s})$, $V = \text{col}(V_s, V_l)$, $C_s = \text{diag}(C_1, \dots, C_{n_s})$, and $P_s = \text{col}(P_1, \dots, P_{n_s})$. Note that

$$P_s = [V_s] I_s = [V_s] (Y_{ss} V_s + Y_{sl} V_l).$$

Remark 6 (Nonlinear consensus algorithms) To compare the algorithm (5) with other nonlinear consensus algorithms, let us neglect the algebraic constraints and the differentiation between source and load nodes. This allows us to rewrite the algorithm as

$$C \dot{V} = -[V] L_c C^{-1} [V] B \Gamma B^T V.$$

The weighted power mean consensus algorithms of [19,20], on the other hand, can be written as $[W] \dot{V} = [V]^{1-r} B \Gamma B^T V$, where W is vector of weights satisfying $\mathbb{1}^T W = 0$ and $r \in \mathbb{R}$. In the special case $r = 0$, we get

$$[W] \dot{V} = [V] B \Gamma B^T V,$$

which is known to converge to the consensus value $V_1^{w_1} \dots V_n^{w_n}$. The analysis is based on the Lyapunov function $\sum_{i=1}^n w_i V_i - \prod_{i=1}^n V_i^{w_i}$.

The nonlinear power consensus algorithm is different in that it uses another layer of averaging in addition to the averaging induced by the physical network. This, and the algebraic constraints, requires a different analysis based on physically inspired Lyapunov functions.

Example 1 Consider a circuit with two sources and a constant impedance load as in Fig. 2. In this case, the current-potential relation (1) is

$$\begin{bmatrix} I_s \\ I_l \end{bmatrix} = B \Gamma B^T V = \begin{bmatrix} \gamma_1 & 0 & -\gamma_1 \\ 0 & \gamma_2 & -\gamma_2 \\ -\gamma_1 & -\gamma_2 & \gamma_1 + \gamma_2 \end{bmatrix} \begin{bmatrix} V_s \\ V_l \end{bmatrix},$$

with $I_s = \text{col}(I_1, I_2)$, $V_s = \text{col}(V_1, V_2)$, $I_l = I_3$, $V_l = V_3$, $\gamma_i = r_i^{-1}$, $i = 1, 2$, and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

The equations of system (10) are in this case

$$\begin{aligned} C_1 \dot{V}_1 &= -V_1 (\gamma_2 C_2^{-1} V_2 (V_2 - V_3) - \gamma_1 C_1^{-1} V_1 (V_1 - V_3)) \\ C_2 \dot{V}_2 &= -V_2 (\gamma_1 C_1^{-1} V_1 (V_1 - V_3) - \gamma_2 C_2^{-1} V_2 (V_2 - V_3)) \\ -\gamma_3 V_3 &= -\gamma_1 (V_1 - V_3) - \gamma_2 (V_2 - V_3), \end{aligned}$$

having used $I_l = I_3 = \gamma_3 V_3 = r_3^{-1} V_3$.

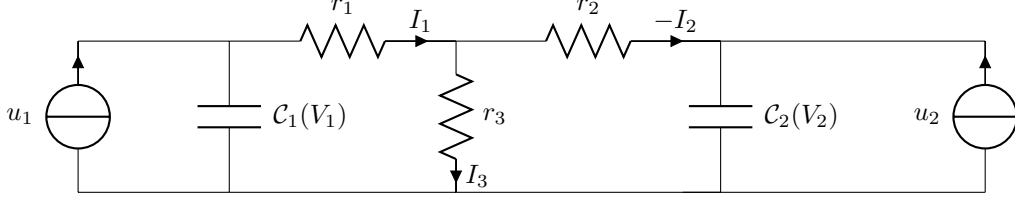


Fig. 2. Circuit considered in Example 1.

4 Power consensus algorithm with constant current loads

In this section we consider constant current loads and analyse the system

$$\begin{bmatrix} C_s \dot{V}_s \\ -I_l^* \end{bmatrix} = - \begin{bmatrix} [V_s] L_c C_s^{-1} P_s \\ B_l \Gamma B^T V \end{bmatrix}. \quad (11)$$

We start by studying its equilibria, namely the set of points $V \in \mathbb{R}_{>0}^n$ which satisfy

$$\begin{bmatrix} 0 \\ -I_l^* \end{bmatrix} = - \begin{bmatrix} [V_s] L_c C_s^{-1} P_s \\ B_l \Gamma B^T V \end{bmatrix} \quad (12)$$

holds.

4.1 Equilibria

Lemma 1 *The set of equilibria of system (11) is*

$$\mathcal{E}_I = \{V \in \mathbb{R}_{>0}^n : \mathcal{P}_I(V_s) = 0, V_l = Y_{ll}^{-1} I_l^* - Y_{ll}^{-1} Y_{ls} V_s\}$$

where $\mathcal{P}_I(V_s)$ depicts the power balance over the sources

$$\mathcal{P}_I(V_s) = \underbrace{[V_s] L_{red} V_s}_{\text{network dissipation}} + \underbrace{[V_s] Y_{sl} Y_{ll}^{-1} I_l^*}_{\text{load demands}} - \underbrace{P_s}_{\text{source injections}},$$

$L_{red} = Y_{ss} - Y_{sl} Y_{ll}^{-1} Y_{ls}$ is known as the Kron-reduced conductance matrix (reduced to the source nodes \mathcal{V}_s), $Y_{sl} Y_{ll}^{-1} I_l^*$ is the mapping of the constant current loads I_l^* to the source buses in the Kron-reduced network [21], and P_s is vector of power injections by the sources written for $V \in \mathcal{E}_I$ as

$$P_s = -C_s \mathbb{1} \frac{\mathbb{1}^T I_l^*}{\mathbb{1}^T [V_s]^{-1} C_s \mathbb{1}} =: C_s \mathbb{1} p_s^*. \quad (13)$$

The steady-state injections (13) achieve indeed power sharing in steady state, and the ultimate power value p_s^* to which the source power injections converge (in a proportional fashion according to the coefficients C_i , $i \in \mathcal{V}_s$) is the total current demand divided by the weighted

sum of the steady state source voltages. The latter values and those of the load voltages are interestingly entangled by the equations $\mathcal{P}_I(V_s) = 0$, $V_l = Y_{ll}^{-1} I_l^* - Y_{ll}^{-1} Y_{ls} V_s$.

PROOF. Let V be an equilibrium of (11), that is let $V \in \mathbb{R}_{>0}^n$ satisfy (12). From the first equation, namely $0 = [V_s] L_c C_s^{-1} P_s$, it immediately follows that $P_s = C_s \mathbb{1} p_s^*$ for some scalar p_s^* . The latter together with the second equation rewrites as

$$\begin{bmatrix} C_s \mathbb{1} p_s^* \\ I_l^* \end{bmatrix} = \begin{bmatrix} [V_s] B_s \Gamma B^T V \\ B_l \Gamma B^T V \end{bmatrix}.$$

Left-multiplying both sides of the identity above by $[\mathbb{1}_{n_s}^T [V_s]^{-1} \mathbb{1}_{n_l}^T]$, it is obtained that

$$\mathbb{1}_{n_s}^T [V_s]^{-1} C_s \mathbb{1}_{n_s} p_s^* + \mathbb{1}_{n_l}^T I_l^* = 0,$$

and solving for p_s^* ,

$$p_s^* = - \frac{\mathbb{1}_{n_l}^T I_l^*}{\mathbb{1}_{n_s}^T [V_s]^{-1} C_s \mathbb{1}_{n_s}}. \quad (14)$$

From $I_l^* = B_l \Gamma B^T V$, and solving for V_l , it follows that

$$V_l = -Y_{ll}^{-1} Y_{ls} V_s + Y_{ll}^{-1} I_l^*, \quad (15)$$

which replaced in $P_s = C_s \mathbb{1} p_s^*$ returns

$$Y_{ss} V_s + Y_{sl} (-Y_{ll}^{-1} Y_{ls} V_s + Y_{ll}^{-1} I_l^*) = [V_s]^{-1} C_s \mathbb{1} p_s^*,$$

or, rearranging the terms,

$$L_{red} V_s + Y_{sl} Y_{ll}^{-1} I_l^* - [V_s]^{-1} C_s \mathbb{1} p_s^* = 0,$$

which, left-multiplying by $[V_s]$ and bearing in mind (14), is precisely $\mathcal{P}_I(V_s) = 0$. The latter and (15) show that $V \in \mathcal{E}_I$.

Conversely, let $V \in \mathcal{E}_I$. Then the equation $I_l^* = B_l \Gamma B^T V$ in (12) is trivially satisfied. From $\mathcal{P}_I(V_s) = 0$, $I_l^* = B_l \Gamma B^T V$ written as (15), and going backwards

through the passages above, we arrive at

$$Y_{ss}V_s + Y_{sl}V_l = [V_s]^{-1}C_s\mathbb{1}p_s^*,$$

or equivalently at $[V_s]B_s\Gamma B^TV = C_s\mathbb{1}p_s^*$. Hence, the power vector $P_s = [V_s]B_s\Gamma B^TV$ satisfies $L_cC_s^{-1}P_s = 0$ that is the first equation in (12), thus showing that $V \in \mathcal{E}_I$ implies the fulfilment of (12).

We make the standing assumption that equilibria exist:

Assumption 2 $\mathcal{E}_I \neq \emptyset$.

Remark 7 (Solvability of $\mathcal{P}_I(V_s) = 0$) *The analytical investigation of the existence of the equilibria \mathcal{E}_I is deferred to a future research. This is a topic of interest on its own and similar problems have been dealt with in recent work about the solvability of reactive power flow equations [22,8,9,23]. For instance, the problem in [23] boils down to the solution of quadratic algebraic equations of the form $[V_l]Y_{ll}V_l - [V_l]Y_{ll}V_l^* + Q_l = 0$, where Q_l is the vector of constant power load demands and V_l^* is the so called vector of open circuit voltages (again constant). Although similarities between these equations and the equations $\mathcal{P}_I(V_s) = 0$ could be useful to investigate the nature of the set \mathcal{E}_I , the non-quadratic nature of $\mathcal{P}_I(V_s) = 0$ might pose additional challenges. Additional insights could come from the convex relaxation of the DC power flow equations in the context of optimal power dispatch [10].*

Example 2 *Consider the case of two sources ($n_s = 2$) and one load ($n_l = 1$) as in Example 1. The equations $\mathcal{P}_I(V_s) = 0$ are in this case*

$$\begin{aligned} \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}V_1(V_1 - V_2) - \frac{\gamma_1}{\gamma_1 + \gamma_2}V_1I_l^* + I_l^*\frac{V_1V_2}{V_{s1} + V_2} &= 0 \\ \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}V_1(V_2 - V_1) - \frac{\gamma_2}{\gamma_1 + \gamma_2}V_2I_l^* + I_l^*\frac{V_1V_2}{V_1 + V_2} &= 0. \end{aligned}$$

We study solutions to the algebraic equations on the curve $V_1V_2 =: c$. The reason for this choice will become clear in Subsection 4.3. On such a curve, the equations simplify as

$$\begin{aligned} V_{s1}^4 - r_2I_l^*V_{s1}^3 + cr_1I_l^*V_{s1} - c^2 &= 0 \\ V_{s2}^4 - r_1I_l^*V_{s2}^3 + cr_2I_l^*V_{s2} - c^2 &= 0, \end{aligned} \quad (16)$$

where $r_i = \gamma_i^{-1}$, $i = 1, 2$ (the resistance of the transmission line i connecting the source i to the load).

These are two independent quartic functions for which an analytic, although involved, expressions of the solutions exist according to the Ferrari-Cardano's formula. The solutions V_1, V_2 to (16) as functions of the ratio r_2/r_1 are depicted in Fig. 3. Following [4], the numerical values in (16) were set as follows: $I_l^* = -0.73$ Ampere, $c = 48^2$ Volt², $r_1 = 0.111$ Ohm.

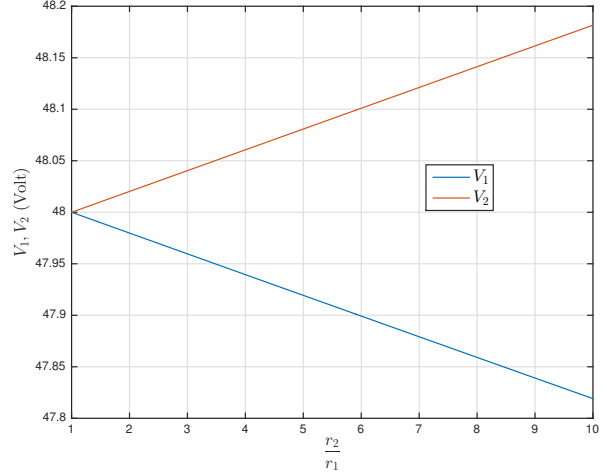


Fig. 3. Solutions V_1, V_2 to (16) as a function of the ratio r_2/r_1 .

Example 3 *Consider the case of n_s sources ($n_s \geq 2$) and one load ($n_l = 1$) connected over a star graph, which is the standard textbook case of a parallel connection of multiple sources feeding a common load bus, as in, e.g., a solar farm. The first n_s nodes correspond to the sources, while the load is associated to the node $n_s + 1$. In this case,*

$$B\Gamma B^T = \begin{bmatrix} \Gamma & -\Gamma\mathbb{1}_{n_s} \\ -\mathbb{1}_{n_s}^T\Gamma & \mathbb{1}^T\Gamma\mathbb{1} \end{bmatrix}$$

and

$$L_{red} = \Gamma - \frac{\Gamma\mathbb{1}_{n_s}\mathbb{1}_{n_s}^T\Gamma}{\mathbb{1}^T\Gamma\mathbb{1}}, \quad Y_{sl}Y_{ll}^{-1} = -\frac{\Gamma\mathbb{1}_{n_s}}{\mathbb{1}^T\Gamma\mathbb{1}}.$$

The system of nonlinear equations $\mathcal{P}_I(V_s) = 0$, with $C_s = \mathbb{1}_2$, writes as

$$[V_s]\Gamma(V_s - \mathbb{1}_{n_s}\frac{\mathbb{1}_{n_s}^T\Gamma V_s}{\mathbb{1}^T\Gamma\mathbb{1}}) - [V_s]\frac{\Gamma\mathbb{1}_{n_s}}{\mathbb{1}^T\Gamma\mathbb{1}}I_l^* + \frac{\mathbb{1}\mathbb{1}^TI_l^*}{\mathbb{1}^T[V_s]^{-1}\mathbb{1}} = 0, \quad (17)$$

or, component-wise,

$$\begin{aligned} V_{s,i} \frac{\gamma_i}{\sum_{j=1}^{n_s} \gamma_j} \sum_{j=1}^{n_s} \gamma_j (V_{s,i} - V_{s,j}) - V_{s,i} \frac{\gamma_i}{\sum_{j=1}^{n_s} \gamma_j} I_l^* + \\ + \frac{C_i I_l^*}{\sum_{j=1}^{n_s} C_j V_j^{-1}} = 0, \quad i \in \mathcal{V}_s. \end{aligned}$$

A solution to these equations such that $V_i = V_j \neq 0$ for all $i, j \in \mathcal{V}_s$, $i \neq j$, exists only if

$$\frac{\gamma_i}{C_i} = \frac{\sum_{j=1}^{n_s} \gamma_j}{\sum_{j=1}^{n_s} C_j}, \quad \forall i \in \mathcal{V}_s,$$

i.e. only if all the products $\frac{\gamma_i}{C_i}$ are the same for all the sources $i \in \mathcal{V}_s$. Conversely, if this condition is fulfilled then $V_i = V_j \neq 0$ for all $i, j \in \mathcal{V}_s$, $i \neq j$, is a solution to the equations. Suppose that there exist other solutions whose components are not all the same, but all different from 0. Then these solutions must necessarily satisfy the equations

$$V_i^2 - \left(\frac{\sum_{j=1}^{n_s} \gamma_j V_j}{\sum_{j=1}^{n_s} \gamma_j} + \frac{I_l^*}{\sum_{j=1}^{n_s} \gamma_j} \right) V_i + \frac{C_i}{\gamma_i} \frac{C_i I_l^*}{\sum_{j=1}^{n_s} C_j V_j^{-1}} = 0, \quad i \in \mathcal{V}_s.$$

Bearing in mind that the ratio $\frac{\gamma_i}{C_i}$ is the same for all i , then these solutions admit the same value, leading to a contradiction. Hence, if $\frac{\gamma_i}{C_i}$ is the same for all the sources $i \in \mathcal{V}_s$, then necessarily $V_i = V_j \neq 0$, for all $i, j \in \mathcal{V}_s$ must be a solution to $\mathcal{P}_I(V_s) = 0$.

4.2 A Lyapunov function

We pursue a Lyapunov-based analysis of the stability of system (11). Inspired by the Lyapunov analysis of reactive power consensus algorithm in [16], we consider the total power dissipated through the network resistors as the first natural Lyapunov candidate for our analysis:

$$U(V) = \frac{1}{2} V^T B \Gamma B^T V. \quad (18)$$

Let $\bar{V} \in \mathcal{E}_I$, and define \bar{P}_s the source power injection corresponding to the equilibrium source voltage \bar{V}_s (see (13)). To cope with the asymmetry in the dynamics of the sources and loads we add to U the term

$$H(V) = -\bar{P}_s^T \ln(V_s),$$

which is the way classical power systems transient stability analysis absorbs constant power injections [18] into a so-called energy function, and we define

$$S(V) = U(V) + H(V). \quad (19)$$

The following Bregman storage function ([16]) is used to center the function S with respect to the equilibrium source voltage \bar{V}_s :

$$\mathcal{S}(V) = S(V) - S(\bar{V}) - \left. \frac{\partial S}{\partial V} \right|_{V=\bar{V}}^T (V - \bar{V}).$$

The next result shows a gradient relation between the dynamics of system (10) and the Bregman storage function above:

Lemma 3 *The following holds:*

$$\begin{bmatrix} L_c C_s^{-1} P_s \\ B_l \Gamma B^T V - I_l^* \end{bmatrix} = \begin{bmatrix} L_c [V_s] C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{S}(V)}{\partial V}.$$

Hence the dynamics (11) rewrites as

$$\begin{bmatrix} C_s \dot{V}_s \\ 0 \end{bmatrix} = - \begin{bmatrix} [V_s] L_c [V_s] C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{S}(V)}{\partial V}.$$

PROOF. We consider the incremental version of U given by

$$\mathcal{U}(V) = \frac{1}{2} (V - \bar{V})^T B \Gamma B^T (V - \bar{V}),$$

whose gradient writes as

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial V} &= B \Gamma B^T (V - \bar{V}) = \begin{bmatrix} I_s \\ I_l \end{bmatrix} - \begin{bmatrix} \bar{I}_s \\ \bar{I}_l \end{bmatrix} \\ &= [V]^{-1} P - [\bar{V}]^{-1} \bar{P} \end{aligned}$$

and the incremental version of H given by

$$\mathcal{H}(V) = -\bar{P}_s^T \ln(V_s) + \bar{P}_s^T [\bar{V}_s]^{-1} (V_s - \bar{V}_s)$$

with gradient

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial V} &= - \begin{bmatrix} [V_s]^{-1} \bar{P}_s \\ 0 \end{bmatrix} + \begin{bmatrix} [\bar{V}_s]^{-1} \bar{P}_s \\ 0 \end{bmatrix} \\ &= -[V]^{-1} \begin{bmatrix} \bar{P}_s \\ 0 \end{bmatrix} + [\bar{V}]^{-1} \begin{bmatrix} \bar{P}_s \\ 0 \end{bmatrix}. \end{aligned}$$

It then follows that $\mathcal{S}(V) = \mathcal{U}(V) + \mathcal{H}(V)$ satisfies

$$\frac{\partial \mathcal{S}}{\partial V} = \begin{bmatrix} [V_s]^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \begin{bmatrix} P_s - \bar{P}_s \\ I_l - \bar{I}_l \end{bmatrix}. \quad (20)$$

In fact

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial V} &= [V]^{-1} \begin{bmatrix} P_s \\ P_l \end{bmatrix} - [\bar{V}]^{-1} \begin{bmatrix} \bar{P}_s \\ \bar{P}_l \end{bmatrix} - \\ &\quad [V]^{-1} \begin{bmatrix} \bar{P}_s \\ 0 \end{bmatrix} + [\bar{V}]^{-1} \begin{bmatrix} \bar{P}_s \\ 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} [V_s]^{-1}(P_s - \bar{P}_s) \\ I_l \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{I}_l \end{bmatrix}$$

where the equality is obtained factoring out $[V]^{-1}$ (respectively, $[\bar{V}]^{-1}$) in the first and third (second and fourth) term of the right-hand side of the first equality and carrying out simple manipulations. Hence, the expression

$$\begin{bmatrix} L_c[V_s]C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{S}(V)}{\partial V}$$

equals

$$\begin{bmatrix} L_c C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \begin{bmatrix} P_s - \bar{P}_s \\ I_l - \bar{I}_l \end{bmatrix},$$

which, bearing in mind that $L_c C_s^{-1} \bar{P}_s = 0$ and $\bar{I}_l = I_l^*$, in turn equals

$$\begin{bmatrix} L_c C_s^{-1} P_s \\ B_l \Gamma B^T V - I_l^* \end{bmatrix}.$$

In view of the dynamics (11), one immediately realizes that

$$\begin{bmatrix} L_c C_s^{-1} P_s \\ B_l \Gamma B^T V - I_l^* \end{bmatrix} = \begin{bmatrix} [V_s]^{-1} C_s \dot{V}_s \\ 0 \end{bmatrix},$$

thus showing the second part of the thesis. This ends the proof.

Remark 8 Notice that one could easily propose different Lyapunov functions whose gradient matches the dynamics of the system. For instance, in the spirit of classic energy functions [18], one may consider the sum of network power dissipation and load power demands as $S(V) = U(V) - \bar{I}_l^T V_l$, which yields the gradient structure

$$\frac{\partial S}{\partial V} = B \Gamma B^T V - [0^T \bar{I}_l^T]^T = \begin{bmatrix} [V_s]^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \begin{bmatrix} P_s \\ I_l - \bar{I}_l \end{bmatrix}.$$

From this gradient formulation (with $K_s = \mathbb{I}_{n_s}$ for the sake of simplicity), one immediately deduces that

$$\begin{bmatrix} [V_s] L_c [V_s] & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial S}{\partial V} = \begin{bmatrix} [V_s] L_c & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \begin{bmatrix} P_s \\ I_l - \bar{I}_l \end{bmatrix}.$$

However, this Lyapunov function candidate misses a key property, namely having a strict local minimum at the equilibrium of interest, as discussed in the next subsection.

4.3 Convergence of solutions

The particular form of the dynamics (11) elucidated in Lemma 3 permits a straightforward analysis of the convergence properties of the solutions.

vergence properties of the solutions.

Theorem 4 The solutions to (11) which originate from any initial condition $V(0)$ belonging to a compact sublevel set Λ_I of \mathcal{S} contained in $\mathbb{R}_{>0}^n$ always remain in Λ_I and converge to the set $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$, where

$$\begin{aligned} \mathcal{V}_I &:= \{ (V_s, V_l) \in \Lambda_I : \\ &V_1^{C_1} \dots V_{n_s}^{C_{n_s}} = V_1^{C_1}(0) \dots V_{n_s}^{C_{n_s}}(0), \\ &V_l = Y_{ll}^{-1} I_l^* - Y_{ll}^{-1} Y_{ls} V_s \}. \end{aligned}$$

Remark 9 (Positive voltages) The proof below shows that a compact sublevel set Λ_I in the positive orthant always exists. Hence the theorem guarantees that voltages that are positive at time $t = 0$ remain so for all $t \geq 0$.

PROOF. Existence and boundedness of solutions. Observe that

$$\frac{\partial^2 \mathcal{S}}{\partial V^2} = B \Gamma B^T + \begin{bmatrix} [V_s]^{-2} \bar{P}_s & 0 \\ 0 & 0 \end{bmatrix}. \quad (21)$$

Hence, the Hessian is positive definite at V if and only if $[V_s]^{-2} \bar{P}_s > 0$.

Let \bar{V} be an equilibrium of the system, i.e. $\bar{V} \in \mathcal{E}_I$. If $I_l^* \in \mathbb{R}_{<0}^{n_l}$, then $[\bar{V}_s]^{-2} \bar{P}_s > 0$, and the storage function \mathcal{S} has an isolated minimum at the equilibrium \bar{V} . Then there exists a compact sublevel set Λ_I of \mathcal{S} around the equilibrium \bar{V} contained in the positive orthant. Consider now solutions to (11) which originate in this level set. These solutions locally exist by the regularity of the algebraic equations. In fact, the algebraic constraint is such that

$$g(V_s, V_l) := Y_{ls}^T V_s + Y_{ll} V_l - I_l^* = 0,$$

and therefore

$$\frac{\partial g}{\partial V_l} = Y_{ll}.$$

Being a principal minor of the Laplacian matrix, the matrix Y_{ll} is positive definite. Hence nonsingularity of $\frac{\partial g}{\partial V_l}$ and therefore regularity of the algebraic condition holds.

When computed along these solutions, $\mathcal{S}(V)$ satisfies

$$\dot{\mathcal{S}}(V) = \frac{\partial \mathcal{S}}{\partial V_s} \dot{V}_s + \frac{\partial \mathcal{S}}{\partial V_l} \dot{V}_l.$$

Notice that

$$\frac{\partial \mathcal{S}}{\partial V_l} = I_l - I_l^* = 0$$

because of the algebraic equation. Hence

$$\dot{\mathcal{S}}(V) = \frac{\partial \mathcal{S}}{\partial V_s}^T \dot{V}_s = -\frac{\partial \mathcal{S}}{\partial V_s}^T C_s^{-1} [V_s] L_c C_s^{-1} \frac{\partial \mathcal{S}}{\partial V_s}, \quad (22)$$

which shows that $\mathcal{S}(V(t))$ is a decreasing function of time. By the compactness of the sublevel set around \bar{V} , the solutions are bounded, exist and belong to Λ_I for all times.

Convergence. Exploiting the regularity of the algebraic equation, the DAE system can be reduced to an ODE system and then the standard LaSalle invariance principle for ODE can be used to infer convergence. We argue as follows.

Any solution (V_s, V_l) to the DAE system (11) is such that its component V_s is a solution to the system of ODE

$$\dot{V}_s = -C_s^{-1} [V_s] L_c [V_s] C_s^{-1} (L_{red} V_s + Y_{sl} Y_{ll}^{-1} I_l^*), \quad (23)$$

whereas $V_l = -Y_{ll}^{-1} Y_{ls} V_s + Y_{ll}^{-1} I_l^*$. Let us set

$$V_l(V_s) := -Y_{ll}^{-1} Y_{ls} V_s + Y_{ll}^{-1} I_l^* \quad (24)$$

to remark the dependence of V_l on V_s .

Define

$$\mathcal{T}(V_s) := \mathcal{S}(V_s, V_l(V_s)) \quad (25)$$

It can be checked that

$$\begin{aligned} \dot{\mathcal{T}}(V_s) &= (P_s - \bar{P}_s)^T [V_s]^{-1} \dot{V}_s \\ &= -(P_s - \bar{P}_s)^T C_s^{-1} L_c C_s^{-1} P_s \\ &= -P_s^T C_s^{-1} L_c C_s^{-1} P_s. \end{aligned} \quad (26)$$

Since V_s is bounded, then the standard La Salle invariance principle for ODE yields convergence of V_s to the largest invariant set where $L_c C_s^{-1} P_s = 0$. As $V_l = -Y_{ll}^{-1} Y_{ls} V_s + Y_{ll}^{-1} I_l^*$, the latter and $L_c C_s^{-1} P_s = 0$ yield that $(V_s, V_l) \in \mathcal{E}_I$ (see Lemma 1). Since (V_s, V_l) is a solution to (11) that remains in Λ_I , convergence to the set $\mathcal{E}_I \cap \Lambda_I$ is inferred. Moreover, the quantity $V_1 \cdot \dots \cdot V_{n_s}$ is conserved, namely $V_1(t) \cdot \dots \cdot V_{n_s}(t) = V_1(0) \cdot \dots \cdot V_{n_s}(0)$ for all t . In fact, by (23), $C_s \frac{d}{dt} \ln V_s = -L_c [V_s] C_s^{-1} (L_{red} V_s + Y_{sl} Y_{ll}^{-1} I_l^*)$, and therefore $\frac{d}{dt} \mathbb{1}^T C_s \ln V_s = 0$. The thesis then follows.

Example 4 Consider again the case of two sources ($n_s = 2$) and one load ($n_l = 1$) connected in a “T” configuration, as in Example 1. If $C_s = \mathbb{I}_2$, the result above shows that on the convergence set $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$, $V_1 V_2 = V_1(0) V_2(0) =: c$ for all $t \geq 0$. Hence, as discussed in Example 2, any point on $\mathcal{E}_I \cap \mathcal{V}_I$ is a solution to the quartic equations (16). In the case $r_1 = r_2 = r$ (identical transmission lines), the expression of the (real

and positive) solution to the quartic equations takes on a particularly simple form, namely

$$V_1 = V_2 = \sqrt{c} = \sqrt{V_{s1}(0) V_{s2}(0)}.$$

It follows that any point on $\mathcal{E}_I \cap \mathcal{V}_I$ is such that each source voltage is the geometric mean of the initial voltage sources. Accordingly, we have for the load voltages that

$$V_l = Y_{ll}^{-1} I_l^* - Y_{ll}^{-1} Y_{ls} V_s = \frac{r}{2} I_l^* + \sqrt{V_{s1}(0) V_{s2}(0)}.$$

The example above shows that the convergence set is made of points parametrised with respect to the initial voltage sources of the system. This actually holds in general and can be formalised as follows.

Corollary 1 The solutions to (11) which originate from any initial condition $V(0)$ belonging to a sublevel set Λ_I of \mathcal{S} contained in $\mathbb{R}_{>0}^n$ always remain in Λ_I and converge to an asymptotically stable equilibrium belonging to $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$.

PROOF. From the proof of Theorem 4, it is known that any solution V_s of the ODE is bounded. By Birckhoff’s Lemma ([24, Lemma 3.1]) the positive limit set $\Omega(V_s)$ associated with a solution $V_s(t)$ is non-empty compact and invariant. Moreover, it is contained in $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$. We would like to prove that $\Omega(V_s)$ is a singleton. To this end, and similarly to [16] we appeal to [25, Proposition 4.7], which states that if the positive limit set $\Omega(V_s)$ contains a Lyapunov stable equilibrium \bar{V}_s , then $\Omega(V_s) = \{\bar{V}_s\}$. To see this first notice that \bar{V}_s being in $\Omega(V_s)$ and hence in $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$, it is indeed an equilibrium of the system. Thus, following (25), one can construct a shifted storage functions $\mathcal{T}(V_s)$ associated to \bar{V}_s . The explicit expression of $\mathcal{T}(V_s)$ is given by

$$\begin{aligned} \mathcal{T}(V_s) &= \frac{1}{2} \begin{bmatrix} V_s - \bar{V}_s \\ V_l(V_s) - V_l(\bar{V}_s) \end{bmatrix}^T B \Gamma B^T \begin{bmatrix} V_s - \bar{V}_s \\ V_l(V_s) - V_l(\bar{V}_s) \end{bmatrix} \\ &\quad - \bar{P}_s^T \ln(V_s) + \bar{P}_s^T \ln(\bar{V}_s) + \bar{P}_s^T [\bar{V}_s]^{-1} (V_s - \bar{V}_s) \end{aligned}$$

where $V_l(V_s)$ is as in (24). The gradient of $\mathcal{T}(V_s)$ is

$$\begin{aligned} \frac{\partial \mathcal{T}}{\partial V_s} &= (Y_{ss} + Y_{sl} \frac{\partial V_l(V_s)}{\partial V_s})^T (V_s - \bar{V}_s) + \\ &\quad (Y_{ls} + Y_{ll} \frac{\partial V_l(V_s)}{\partial V_s})^T (V_l(V_s) - V_l(\bar{V}_s)) \\ &\quad - [V_s]^{-1} \bar{P}_s + [\bar{V}_s]^{-1} \bar{P}_s \\ &= (Y_{ss} - Y_{sl} Y_{ll}^{-1} Y_{ls})^T (V_s - \bar{V}_s) + \end{aligned}$$

$$\begin{aligned}
& (Y_{ls} - Y_{ll}Y_{ll}^{-1}Y_{ls})^T(-Y_{ll}^{-1}Y_{ls}V_s + Y_{ll}^{-1}I_l^* - V_l(\bar{V}_s)) \\
& - [V_s]^{-1}\bar{P}_s + [\bar{V}_s]^{-1}\bar{P}_s \\
& = (Y_{ss} - Y_{sl}Y_{ll}^{-1}Y_{ls})^T(V_s - \bar{V}_s) \\
& - [V_s]^{-1}\bar{P}_s + [\bar{V}_s]^{-1}\bar{P}_s.
\end{aligned}$$

The gradient $\frac{\partial \mathcal{T}}{\partial V_s}$ vanishes if $V_s = \bar{V}_s$. Bearing in mind that $Y_{ss} - Y_{sl}Y_{ll}^{-1}Y_{ls} = L_{red}$, it follows that

$$\frac{\partial^2 \mathcal{T}}{\partial V_s^2} = L_{red} + [V_s]^{-2}\bar{P}_s.$$

This shows that \mathcal{T} has a strict local minimum at \bar{V}_s . By (26), $\dot{\mathcal{T}} \leq 0$, and these two properties (properness and nonnegative time derivative) show that \bar{V}_s is a Lyapunov stable equilibrium, and therefore $\Omega(V_s) = \{\bar{V}_s\}$, thus proving that the solution $V_s(t)$ converges to an equilibrium point. Because $V_s(t)$ is the V_s component of the solution to the DAE, and since $V_l(t)$ satisfies (24) we also see that the solution $(V_s(t), V_l(t))$ of the DAE converges to a point in $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$. Since this equilibrium point is Lyapunov stable by (21) and (22), it is also shown asymptotic stability of the convergence point. This ends the proof.

Remark 10 (Capacitors at the loads) *If loads are interconnected to the network via capacitors, the load equations are modified as*

$$C_l \dot{V}_l = -I_l^* + B_l \Gamma B^T V.$$

Notice that the equilibria of the system remain the same. Bearing in mind (21), the load dynamics can be written as

$$C_l \dot{V}_l = -\frac{\partial \mathcal{S}}{\partial V_l}.$$

It follows that

$$\dot{\mathcal{S}} = -\frac{\partial \mathcal{S}}{\partial V_s}^T C_s^{-1} [V_s] L_c C_s^{-1} \frac{\partial \mathcal{S}}{\partial V_s} - \frac{\partial \mathcal{S}}{\partial V_l}^T C_l^{-1} \frac{\partial \mathcal{S}}{\partial V_l},$$

and one can infer convergence to the set $\mathcal{E}_I \cap \Lambda_I \cap \mathcal{V}_I$ similarly as for the differential-algebraic model.

Remark 11 (Constant voltage loads) *Similarly as in [3, Remark 3.3], one can consider the case of all load voltages having constant voltages V_l (see [3] for a discussion on this load condition). In this case, the dynamics rewrite as*

$$\begin{aligned}
C_s \dot{V}_s &= -[V_s] L_c C_s^{-1} [V_s] (Y_{ss} V_s + Y_{sl} V_l) \\
-I_l &= Y_{ls} V_s + Y_{ll} V_l.
\end{aligned} \tag{27}$$

The second equation implies that a controller adjust the current load I_l depending on V_s to maintain the value

of the load voltage at the constant level V_l . Hence, from the point of view of the system's stability, the only relevant equation in (27) is the first one, which can be viewed as a system of nonlinear ordinary differential equations driven by the constant external forcing term V_l . We study the stability of the system using a slightly modified version of the previous Lyapunov inequality. In fact, since V_l is now constant, it suffice to consider the Lyapunov function \mathcal{S} with \mathcal{U} now given by $\mathcal{U}(V_s) = \frac{1}{2}(V_s - \bar{V}_s)^T Y_{ss} (V_s - \bar{V}_s)$. Note that $\frac{\partial \mathcal{U}}{\partial V_s} = Y_{ss}(V_s - \bar{V}_s) = (Y_{ss} V_s + Y_{sl} V_l) - (Y_{ss} \bar{V}_s + Y_{sl} V_l) = [V_s]^{-1} P_s - [\bar{V}_s]^{-1} \bar{P}_s$, where as before $L_c C_s^{-1} \bar{P}_s = 0$. Bearing in mind the definition of \mathcal{H} used in the definition of \mathcal{S} , namely $\mathcal{H}(V) = -\bar{P}_s^T \ln(V_s) + \bar{P}_s^T \ln(\bar{V}_s) + \bar{P}_s^T [\bar{V}_s]^{-1} (V_s - \bar{V}_s)$, we obtain that $\frac{\partial \mathcal{S}}{\partial V_s} = [V_s]^{-1} (P_s - \bar{P}_s)$. Since $C_s \dot{V}_s = -[V_s] L_c C_s^{-1} P_s$, then also $C_s \dot{V}_s = -[V_s] L_c C_s^{-1} [V_s] \frac{\partial \mathcal{S}}{\partial V_s}$. This expression can be used to characterize the convergence of the solutions of the system (27) similarly as in Theorem 4. Details are omitted.

5 The case of constant impedance loads

The previous analysis can be easily extended to the case of constant impedance loads. The model in this case is

$$\begin{bmatrix} C_s \dot{V}_s \\ Y_l^* V_l \end{bmatrix} = - \begin{bmatrix} [V_s] L_c C_s^{-1} [V_s] I_s \\ I_l \end{bmatrix} \tag{28}$$

The equilibria of the system can be characterized similarly as in Lemma 1:

Lemma 5 *The set of equilibria of system (28) is*

$$\mathcal{E}_Z = \{V \in \mathbb{R}_{>0}^n : \mathcal{P}_Z(V_s) = 0, V_l = -(Y_l^* + Y_{ll})^{-1} Y_{ls} V_s\}$$

where $\mathcal{P}_Z(V_s)$ depicts the power balance over the sources

$$\mathcal{P}_Z(V_s) = \underbrace{[V_s] \hat{L}_{red} V_s}_{\text{network \& load dissipation}} - \underbrace{P_s}_{\text{source injections}},$$

$\hat{L}_{red} = Y_{ss} - Y_{sl}(Y_{ll} + Y_l^)^{-1} Y_{ls}$ is the Kron-reduced conductance matrix (reduced to the sources) that absorbed the constant impedance loads, and P_s is vector of power injections by the sources written for for $V \in \mathcal{E}_Z$ as*

$$P_s = -C_s \mathbb{1} \frac{\mathbb{1}^T Y_l^* (Y_l^* + Y_{ll})^{-1} Y_{ls} V_s}{\mathbb{1}^T [V_s]^{-1} C_s^{-1} \mathbb{1}} =: C_s \mathbb{1} p_s^*. \tag{29}$$

The proof follows the lines of the proof of Lemma 1 and is therefore omitted. Similarly to the case of constant current case, it is assumed throughout the section that the equilibrium set is non-empty: $\mathcal{E}_Z \neq \emptyset$.

The algebraic equation writes in this case as $\mathbb{0} = (B_l \Gamma B^T + [0 \ Y_l^*])V = Y_{ls}V_s + (Y_{ll} + Y_l^*)V_l$, which resembles the algebraic equation in the case of constant current load with a modified admittance submatrix Y_{ll} , namely $Y_{ll} + Y_l^*$, and with $I_l^* = 0$. This suggests to use as a Lyapunov function a variation of the one for the constant current load, namely $N(V) = J(V) + H(V)$, where J replaces U and takes into account the expression of the dissipated power through the impedance loads as

$$J(V) = \frac{1}{2}V^T(B\Gamma B^T + \begin{bmatrix} 0 & 0 \\ 0 & Y_l^* \end{bmatrix})V. \quad (30)$$

An immediate consequence of this choice is the following:

Lemma 6 *The Bregman function $\mathcal{N}(V) = N(V) - N(\bar{V}) - \frac{\partial N}{\partial V} \Big|_{V=\bar{V}}^T (V - \bar{V})$ satisfies*

$$\begin{bmatrix} L_c C_s^{-1} P_s \\ B_l \Gamma B^T V + Y_l^* V_l \end{bmatrix} = \begin{bmatrix} L_c[V_s]C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{N}}{\partial V}.$$

Hence the dynamics (28) rewrites as

$$\begin{bmatrix} C_s \dot{V}_s \\ 0 \end{bmatrix} = - \begin{bmatrix} [V_s]L_c[V_s]C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{N}}{\partial V}.$$

PROOF. First note that $\frac{\partial \mathcal{N}}{\partial V_s}$ equals $\frac{\partial S}{\partial V_s}$, this is the first identity in the statement immediately descends from Lemma 3. Second, $\frac{\partial \mathcal{N}}{\partial V_l} = B_l \Gamma B^T (V - \bar{V}) + Y_l^* (V_l - \bar{V}_l)$, and the second identity holds because at the equilibrium \bar{V} , $\mathbb{0} = B_l \Gamma B^T \bar{V} + Y_l^* \bar{V}_l$. Hence the same Lyapunov function can be used for the analysis. Details are omitted.

Next we show attractivity of the equilibria:

Theorem 7 *The solutions to (28) which originate from a sublevel set Λ_Z of \mathcal{N} contained in $\mathbb{R}_{>0}^n$ always remain in Λ_Z and converge to the set $\mathcal{E}_Z \cap \Lambda_Z \cap \mathcal{V}_Z$, where*

$$\begin{aligned} \mathcal{V}_Z &:= \{ (V_s, V_l) \in \Lambda_Z : \\ &V_1^{C_1} \dots V_{n_s}^{C_{n_s}} = V_1^{C_1}(0) \dots V_{n_s}^{C_{n_s}}(0), \\ &V_l = -(Y_l^* + Y_{ll})^{-1} Y_{ls} V_s \}. \end{aligned}$$

PROOF. The property $Y_{ll} + Y_l^* > 0$ guarantees strict convexity of \mathcal{N} on $\mathbb{R}_{>0}^n$, and therefore existence of compact sublevel sets Λ_Z of \mathcal{N} in $\mathbb{R}_{>0}^n$. Under the same

property the algebraic condition is solvable and $V_l(V_s) = -(Y_l^* + Y_{ll})^{-1} Y_{ls} V_s$. We consider the reduced ODE system $\dot{V}_s = -C_s^{-1}[V_s]L_c C_s^{-1}[V_s](Y_{ls} - Y_{ll}(Y_l^* + Y_{ll})^{-1} Y_{ls})V_s$ and consider the evolution of $\mathcal{N}(V_s, V_l(V_s))$ along its solutions. We obtain that

$$\begin{aligned} \frac{d}{dt} \mathcal{N}(V_s, V_l(V_s)) &= \frac{\partial \mathcal{N}(V_s, V_l)}{\partial V_s} \Big|_{V_l=V_l(V_s)}^T \dot{V}_s \\ &= \frac{\partial S(V_s, V_l)}{\partial V_s} \Big|_{V_l=V_l(V_s)}^T \dot{V}_s \\ &= -P_s^T C_s^{-1} L_c C_s^{-1} P_s, \end{aligned}$$

as in the proof of Theorem 4. From this point on, the proof follows the same arguments therein and is therefore omitted.

Remark 12 (ZI loads) *The same analysis can be extended to the case in which loads are parallel connections of a constant current and a constant impedance load. In this case, the algebraic equation is $I_l^* - Y_l^* V_l = B_l \Gamma B^T V$. From the proof of Lemma 6 it is known that $\frac{\partial \mathcal{N}}{\partial V_l} = B_l \Gamma B^T (V - \bar{V}) + Y_l^* (V_l - \bar{V}_l)$. Notice now that \bar{V} satisfies $I_l^* - Y_l^* \bar{V}_l = B_l \Gamma B^T \bar{V}$, from which we have $\frac{\partial \mathcal{N}}{\partial V_l} = B_l \Gamma B^T V + Y_l^* V_l - I_l^*$. Hence, the equality in Lemma 6 can be replaced by*

$$\begin{bmatrix} L_c C_s^{-1} P_s \\ B_l \Gamma B^T V + Y_l^* V_l - I_l^* \end{bmatrix} = \begin{bmatrix} L_c[V_s]C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{N}}{\partial V}.$$

6 The case of constant power loads

In this section we focus on a different load model, namely we consider a vector $P_l^* \in \mathbb{R}_{>0}^{n_l}$ of constant power loads giving rise to the closed-loop system

$$\begin{bmatrix} C_s \dot{V}_s \\ -P_l^* \end{bmatrix} = - \begin{bmatrix} [V_s]L_c C_s^{-1} P_s \\ [V_l]I_l \end{bmatrix}. \quad (31)$$

The set of equilibria of system (31) can be derived similarly as in Lemma 1. *Mutatis mutandis*, the following holds:

Lemma 8 *The set of equilibria of system (31) is*

$$\begin{aligned} \mathcal{E}_P &= \{V \in \mathbb{R}_{>0}^n : \mathcal{P}_P(V_s, V_l) = 0, \\ &V_l = Y_{ll}^{-1}[V_l]^{-1} P_l^* - Y_{ll}^{-1} Y_{ls} V_s \} \end{aligned} \quad (32)$$

where $\mathcal{P}_P(V_s)$ depicts the power balance over the sources

$$\mathcal{P}_P(V_s, V_l) = \underbrace{[V_s]L_{red}V_s}_{\text{network dissipation}} + \underbrace{[V_s]Y_{sl}Y_{ll}^{-1}[V_l]^{-1}P_l^*}_{\text{load demands}} - \underbrace{P_s}_{\text{source injections}}, \quad (33)$$

$L_{red} = Y_{ss} - Y_{sl}Y_{ll}^{-1}Y_{ls}$ is again the Kron-reduced conductance matrix, $[V_s]Y_{sl}Y_{ll}^{-1}[V_l]^{-1}P_l^*$ is the mapping of the constant power loads P_l^* to the source buses in the Kron-reduced network, and P_s is vector of power injections by the sources written for $V \in \mathcal{E}_P$ as

$$P_s = -C_s \mathbb{1} \frac{\mathbb{1}^T [V_l]^{-1} P_l^*}{\mathbb{1}^T [V_s]^{-1} C_s \mathbb{1}} =: \mathbb{1} p_s^*. \quad (34)$$

Remark 9 (Average voltage inequality) By definition $P = [V]B\Gamma B^T V$, from which we obtain the total (or average) power inequality $\mathbb{1}^T P_s + \mathbb{1}^T P_l^* = V^T B\Gamma B^T V \geq 0$. Bearing in mind (34), the inequality holds if and only if

$$-\mathbb{1}^T C_s \mathbb{1} \frac{\mathbb{1}^T [V_l]^{-1} P_l^*}{\mathbb{1}^T [V_s]^{-1} C_s \mathbb{1}} + \mathbb{1}^T P_l^* \geq 0,$$

or equivalently,

$$\frac{\sum_{i \in \mathcal{V}_l} \frac{P_{l,i}^*}{V_i}}{\sum_{i \in \mathcal{V}_l} \frac{P_{l,i}^*}{V_i}} \leq \frac{\sum_{i \in \mathcal{V}_s} \frac{C_i}{V_i}}{\sum_{i \in \mathcal{V}_s} \frac{C_i}{V_i}}.$$

This inequality can be also expressed as

$$\sum_{i \in \mathcal{V}_l} \frac{a_i}{V_i} \geq \sum_{i \in \mathcal{V}_s} \frac{b_i}{V_i} \quad (35)$$

$a_i = P_{l,i}^* / \sum_{i \in \mathcal{V}_l} P_{l,i}^*$ and $b_i = C_i / \sum_{i \in \mathcal{V}_s} C_i$, which relates a convex combination of the inverses of the voltages at the loads, with a convex combination of the inverses of the voltages at the sources, and represents another relation between V_s, V_l in addition to those in (32). Thus, aside from the ‘‘average power inequality’’ $\mathbb{1}^T P_s \geq -\mathbb{1}^T P_l^*$, an ‘‘average voltage inequality’’ (35) also holds.

Assumption 2 is still standing, with \mathcal{E} now replaced by \mathcal{E}_P defined as in (32).

The Lyapunov function to investigate the dynamical properties of (31) stems from suitable modification of the function $\mathcal{S}(V)$ used to study the case of constant current loads. We introduce the function

$$R(V) = S(V) + K(V)$$

where $S(V)$ is defined as in (19) and $K(V)$ accounts for constant power loads as in [18]:

$$K(V) = -P_l^{*T} \ln(V_l).$$

The corresponding Bregman storage function is

$$\mathcal{R}(V) = \mathcal{S}(V) - P_l^{*T} \ln(V_l) + P_l^{*T} \ln(\bar{V}_l) + P_l^{*T} [\bar{V}_l]^{-1} (V_l - \bar{V}_l).$$

Lemma 10 For $V \in \mathbb{R}_{>0}^n$, the following identity holds:

$$\begin{bmatrix} L_c C_s^{-1} P_s \\ I_l - [V_l]^{-1} P_l^* \end{bmatrix} = \begin{bmatrix} L_c C_s^{-1} [V_s] & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{R}}{\partial V}.$$

Hence the dynamics (31) rewrites as

$$\begin{bmatrix} C_s \dot{V}_s \\ 0 \end{bmatrix} = - \begin{bmatrix} [V_s] L_c [V_s] C_s^{-1} & 0 \\ 0 & \mathbb{I}_{n_l} \end{bmatrix} \frac{\partial \mathcal{R}}{\partial V}.$$

PROOF. Observe that

$$\begin{aligned} \frac{\partial \mathcal{R}(V)}{\partial V} &= \frac{\partial \mathcal{S}(V)}{\partial V} - \begin{bmatrix} 0 \\ [V_l]^{-1} P_l^* - [\bar{V}_l]^{-1} P_l^* \end{bmatrix} \\ &= \begin{bmatrix} [V_s]^{-1} (P_s - \bar{P}_s) \\ I_l - \bar{I}_l \end{bmatrix} - \begin{bmatrix} 0 \\ [V_l]^{-1} P_l^* - [\bar{V}_l]^{-1} P_l^* \end{bmatrix} \\ &= \begin{bmatrix} [V_s]^{-1} (P_s - \bar{P}_s) \\ I_l - [V_l]^{-1} P_l^* \end{bmatrix}, \end{aligned} \quad (36)$$

where (21) was used to derive the second equality, and the algebraic equation at the equilibrium $\bar{V} \in \mathcal{E}_P$, namely the identity $\bar{I}_l = [\bar{V}_l]^{-1} P_l^*$, was used for the third equality. The thesis immediately follows.

The following is the main result of this section:

Theorem 11 Assume that there exists $\bar{V} \in \mathcal{E}_P$ such that

$$Y_{ll} + [\bar{V}_l]^{-2} [P_l^*] > 0. \quad (37)$$

Then there exists a sufficiently small compact sublevel set Λ_P of $\mathcal{R}(V)$ containing \bar{V} and contained in $\mathbb{R}_{>0}^n$ such that any solution to (31) originating from $V(0)$ in Λ_P exists and asymptotically converges to the set $\mathcal{E}_P \cap \Lambda_P \cap \mathcal{V}_P$, where $\mathcal{V}_P := \{(V_s, V_l) \in \Lambda_P : V_1^{C_1} \cdots V_{n_s}^{C_{n_s}} = V_1^{C_1}(0) \cdots V_{n_s}^{C_{n_s}}(0), [V_l]Y_{ll}V_l + [V_l]Y_{ls}V_s - P_l^* = 0\}$.

PROOF. Consider the algebraic equation $g(V) := I_l - [V_l]^{-1} P_l^* = Y_{ls}V_s + Y_{ll}V_l - [V_l]^{-1} P_l^* = 0$ and its Jacobian $\frac{\partial g}{\partial V_l} = Y_{ll} + [V_l]^{-2} [P_l^*]$. Let $\bar{V} \in \mathcal{E}_P$ be such that the Jacobian evaluated at \bar{V} is positive definite, that is $Y_{ll} + [\bar{V}_l]^{-2} [P_l^*] > 0$. Then there exists a neighbourhood

of \bar{V} where the algebraic equation $[V_l]I_l - P_l^* = 0$ can be solved for V_l . Denote by $V_l = \delta(V_s)$ the resulting map giving V_l as a function of V_s . Hence, in such a neighbourhood existence of solutions to (31) can be locally guaranteed.

In view of Lemma 10, the function $\mathcal{R}(V)$ computed along the solutions to (31) satisfy

$$\dot{\mathcal{R}}(V) = -\frac{\partial \mathcal{R}(V)}{\partial V_s}^T C_s^{-1}[V_s] L_c C_s^{-1}[V_s] \frac{\partial \mathcal{R}(V)}{\partial V_s} - \frac{\partial \mathcal{R}(V)}{\partial V_l}^T \dot{V}_l.$$

Recall that

$$\frac{\partial \mathcal{R}(V)}{\partial V_l} = [V_l]^{-1}([V_l]I_l - P_l^*) = 0,$$

in view of the algebraic equation. Hence, the expression of $\dot{\mathcal{R}}(V)$ simplifies as

$$\dot{\mathcal{R}}(V) = -\frac{\partial \mathcal{R}}{\partial V_s}^T [V_s] C_s^{-1} L_c [V_s] C_s^{-1} \frac{\partial \mathcal{R}}{\partial V_s}, \quad (38)$$

where, by (36),

$$\frac{\partial \mathcal{R}}{\partial V_s} = [V_s]^{-1}([V_s](Y_{ss}V_s + Y_{sl}V_l) - \bar{P}_s)$$

In particular, the Lyapunov function $\mathcal{R}(V)$ cannot increase along the solutions to (31).

Since $\bar{V} \in \mathcal{E}_P$, then \bar{V} is a critical point of \mathcal{R} . Moreover, the Hessian of \mathcal{R} is given by

$$\frac{\partial^2 \mathcal{R}}{\partial V^2} = \begin{bmatrix} [V_s]^{-2}[\bar{P}_s] + Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} + [V_l]^{-2}[P_l^*] \end{bmatrix}. \quad (39)$$

By (37), and since $\bar{P}_s \in \mathbb{R}_{>0}^{n_s}$, the Hessian matrix above is positive definite at \bar{V} , hence the latter is a strict local minimum for \mathcal{R} . Then there exists a compact sublevel set Λ_P of $\mathcal{R}(V)$ containing \bar{V} and contained in $\mathbb{R}_{>0}^n$. Moreover, without loss of generality, and by continuity, it can be assumed that the sublevel set Λ_P is sufficiently small to be contained in the neighbourhood of \bar{V} where the regularity condition (37) is satisfied.

Because of the latter, and as already pointed out above, any solution originating in Λ_P locally exists. Moreover, since the set Λ_P is a compact sublevel set of \mathcal{R} and (38) holds, the solution cannot leave Λ_P and therefore it does not exhibit a finite escape time and exists for all time.

On this positively invariant sublevel set Λ_P , one can solve the algebraic equation $[V_l]I_l = P_l^*$ for V_l , obtain

the solution $V_l = \delta(V_s)$ for some suitable map δ and consider the reduced dynamics

$$\dot{V}_s = -[V_s]L_c[V_s](Y_{ss}V_s + Y_{sl}\delta(V_s)). \quad (40)$$

It was already shown that the solution V_s is bounded. Furthermore, in view of (38), the function $\mathcal{R}(V_s, \delta(V_s))$ along the solution to (40) satisfy

$$\begin{aligned} \frac{d}{dt}\mathcal{R}(V_s, \delta(V_s)) &= \frac{d}{dt}\mathcal{R}(V_s, V_l)|_{V_l=\delta(V_s)} \\ &= \frac{\partial \mathcal{R}(V_s, V_l)|_{V_l=\delta(V_s)}}{\partial V_s}^T \dot{V}_s \\ &= (P_s - \bar{P}_s)^T C_s^{-1} L_c C_s^{-1} (P_s - \bar{P}_s) \\ &= (Y_{ss}V_s + Y_{sl}\delta(V_s))^T [V_s] C_s^{-1} L_c C_s^{-1} [V_s] (Y_{ss}V_s + Y_{sl}\delta(V_s)). \end{aligned}$$

As system (40) is a system of ordinary differential equations, a standard La Salle's invariance principle can be applied to conclude that the solutions originating in Λ_P converge to the largest invariant set where $L_c C_s^{-1}[V_s](Y_{ss}V_s + Y_{sl}\delta(V_s)) = 0$. Bearing in mind that on Λ_P , $V_l = \delta(V_s)$ and that $[V_l]I_l = P_l^*$, it is seen that the points such that $L_c C_s^{-1}[V_s](Y_{ss}V_s + Y_{sl}\delta(V_s)) = 0$ are equilibria of (31), and as such that they belong to $\mathcal{E}_P \cap \Lambda_P$. Furthermore, it is known that any solution to (31) satisfies the invariance property $1^T C_s \ln(V_s(t)) = 0$ for all t . This last observation leads to the thesis.

Example 5 *In the case of a network with two sources and one load arranged in a "T" configuration (Example 1), using the same arguments that led to (16), one finds that $\mathcal{P}_P(V_s, V_l) = 0$, with $\mathcal{P}_P(V_s, V_l)$ as in (33), if and only if*

$$\begin{aligned} V_l V_1^4 - r_2 P_l^* V_1^3 + c r_1 P_l^* V_1 - c^2 V_l &= 0 \\ V_l V_2^4 - r_1 P_l^* V_2^3 + c r_2 P_l^* V_2 - c^2 V_l &= 0. \end{aligned} \quad (41)$$

In the equations above V_l is the value of the voltage load as obtained from the solution of the algebraic equation. Taking V_l as a positive parameter, and under the condition $r_1 = r_2 = r$, one can analytically solve the two quartic functions above, and obtain as in Example 4 that the only positive solution is $V_1 = V_2 = \sqrt{c}$, thus independent of V_l . Replacing in the algebraic equation $0 = V_l B_l \Gamma B^T V = P_l^$, one obtains that*

$$\begin{aligned} 0 &= V_l(-\gamma_1 V_1 - \gamma_2 V_2 + (\gamma_1 + \gamma_2)V_l) - P_l^* \\ &= 2\gamma V_l^2 - 2\gamma\sqrt{c}V_l - P_l^*. \end{aligned}$$

Bearing in mind that $\gamma = r^{-1}$ and $c = V_1(0)V_2(0)$, we conclude that if $V_1(0)V_2(0) + 2rP_l^ > 0$, then there exists a unique positive voltage load solution of the equation*

above and is given by

$$V_l = \frac{\sqrt{V_1(0)V_2(0)}}{2} + \sqrt{\frac{V_1(0)V_2(0)}{4} + \frac{r}{2}P_l^*}.$$

If in addition the point computed above, namely

$$V_1 = V_2 = \sqrt{V_1(0)V_2(0)},$$

$$V_l = \frac{\sqrt{V_1(0)V_2(0)}}{2} + \sqrt{\frac{V_1(0)V_2(0)}{4} + \frac{r}{2}P_l^*},$$

satisfies the regularity condition (37), which in this case writes as $\frac{2}{r} + V_l^{-2}P_l^* > 0$, then it is the only point in the convergence set $\mathcal{E}_P \cap \Lambda_P \cap \mathcal{V}_P$, and Theorem 11 guarantees convergence to this point. Notice that the regularity condition can be rewritten as $V_l^2 + \frac{r}{2}P_l^* > 0$, and since $V_l^2 > V_1(0)V_2(0)$, the condition $V_1(0)V_2(0) + 2rP_l^* > 0$ guarantees that the regularity condition is also satisfied.

Remark 13 (Capacitors at the loads) As in the case of constant current loads (see Remark 10), also in the case of constant power (and constant impedance) loads, it is possible to show that if capacitors are present at the loads then the proposed power consensus algorithm still generates trajectories that converge to the desired convergence set.

Remark 14 (ZIP loads) If loads are given by a parallel combination of constant impedance, current and power loads, then the analysis can be carried out following the previous line of arguments. This can be realized by considering the Lyapunov function given by the combination of those used in the case of constant impedance and constant power loads, namely the function

$$M(V) := J(V) + H(V) + K(V)$$

$$= \frac{1}{2}V^T(B\Gamma B^T + \begin{bmatrix} 0 & 0 \\ 0 & Y_l^* \end{bmatrix})V - \bar{P}_s^T \ln(V_s) - P_l^{*T} \ln(V_l).$$

Then, exploiting the identity $B_l\Gamma B^T \bar{V} = I_l^* - Y_l^* \bar{V}_l - [\bar{V}_l]^{-1}P_l^*$,

$$\frac{\partial \mathcal{M}}{\partial V} = \begin{bmatrix} [V_s]^{-1}P_s \\ B_l\Gamma B^T V - I_l^* + Y_l^* V_l - [V_l]^{-1}P_l^* \end{bmatrix}$$

where \mathcal{M} is the Bregman storage function corresponding to M . Hence the dynamical system with ZIP loads can be

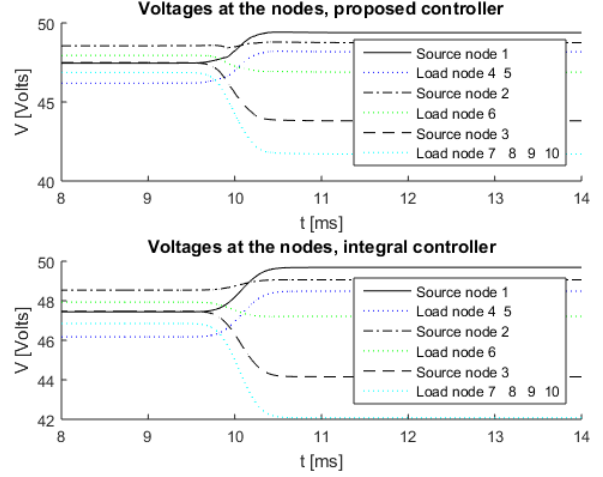


Fig. 4. Voltage plots of the simulation

rewritten as

$$\begin{bmatrix} C_s \dot{V}_s \\ 0 \end{bmatrix} = - \begin{bmatrix} [V_s]L_c C_s^{-1} P_s \\ B_l\Gamma B^T V - I_l^* + Y_l^* V_l - [V_l]^{-1}P_l^* \end{bmatrix}$$

$$= - \begin{bmatrix} [V_s]L_c C_s^{-1} [V_s] & 0 \\ 0 & \mathbb{I}_{n_s} \end{bmatrix} \frac{\partial \mathcal{M}}{\partial V}.$$

7 Simulations

In this section, we present simulation results comparing the proposed control strategy to an averaging-based control method. We use an example network obtained from [4]. The network topology is sketched in Fig. 6, and the physical parameters are given in Table 1.

As in the reference experiment, there are seven constant power loads, five of which are initially turned off and are turned on gradually between 9.5 and 10.5 ms. This means that there is a gradual increase of the total power load from 70 W to 245 W. We simulate both the proposed control strategy (5), and the distributed averaging integral controller (9) for a comparison. The power measured at the source nodes is shown for both control strategies in Fig. 5. As predicted by the analysis, at steady state proportional power sharing is achieved by the power sources in conformity with (6). We also observe that the two controllers perform similarly, only a slight overshoot for the integral controller at the power source 2 can be observed. The voltage evolution both at the sources and at the loads is depicted in Fig. 4.

8 Conclusions

We have proposed controllers for DC microgrids that average power measurement at the sources. The results

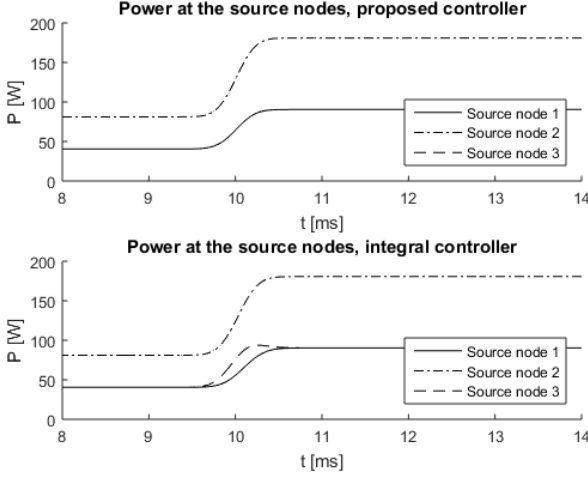


Fig. 5. Power plots of the simulation

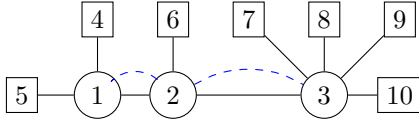


Fig. 6. The node network used for the simulations. Sources are depicted as circles, loads as rectangles. Solid lines denote the transmission lines, while dashed blue lines represent the communication graph used by the controllers.

Parameter	Value
Transmission line weights Γ_i	$6 \times 10^{-1} \Omega$
Capacitance weight C_i , $i = 1, 3$	$4 \times 10^{-2} \sqrt{\text{kgm/s}}$
$i = 2$	$8 \times 10^{-2} \sqrt{\text{kgm/s}}$
Nominal voltage V^*	48 V
Integral controller weights D_i	1×10^{-4}
Load values $-P_l^*$	35 W

Table 1

Simulation parameter values.

apply to network preserved model (systems of DAE) of the microgrid in the presence of ZIP loads. Capacitors at the terminals of the grid that model either Π -models of the transmission lines or power converters components can be included by means of passivity-based analysis.

Many interesting new research directions can be taken. The first one is to consider more complex scenarios such as the inclusion of dynamical transmission lines and dynamical loads. Another one is the extensions of the results of this paper to network preserved AC microgrids. Moreover, although the preservation of the geometric mean of the voltages allows for an estimate of the voltage excursion, no active voltage regulation is present in the proposed scheme. An addition of a voltage controller to the power consensus algorithm is an interest-

ing open problem. It is remarkable that these power consensus algorithms preserve the geometric mean of the voltages. Nonlinear consensus algorithms that converge towards various weighted means (see e.g. [19,20]) are a compelling field of study and establishing a deeper connection of the power consensus algorithms studied in this paper with them is worth an investigation. The power consensus algorithms lead to a new set of power flow equations, whose solvability we did not investigate. The existence of a solution to these equations could be studied starting from recent advances in the existence and approximation of power flow equations ([22,8,23] and references therein). Finally, we would like to analyze the distributed averaging proportional integral controller (9) discussed in Remark 5 similarly to what we have done in this paper with the power consensus algorithm (5). The former enjoys the nice feature of not requiring power measurements and could be an enthralling algorithm to investigate further.

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